N=8 Supersymmetric Quaternionic Mechanics

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Abstract

We construct $N=8$ supersymmetric mechanics with four bosonic end eight fermionic physical degrees of freedom. Starting from the most general $N=4$ superspace action in harmonic superspace for the (**4,8,4**) supermultiplet we find conditions which make it $N=8$ invariant. We introduce in the action Fayet-Iliopoulos terms which give rise to potential terms. We present the action in components and give explicit expressions for the Hamiltonian and Poisson brackets. Finally we discuss the possibility of $N=9$ supersymmetric mechanics.

1 Introduction

One dimensional extended supersymmetry has plenty of features which make it interesting not only due to relations with higher dimensional theories but also as independent theory. The restrictions which extended supersymmetry puts on the geometry of the target space (see e.g. [1]) makes the systems with eight real supercharges mostly interesting among N supersymmetric extended ones. Indeed, $N=8$ supersymmetry is large enough to describe a rather complicated system and, at the same time, it does not put too strong restrictions on the geometry of the bosonic sector.

In the last decade some progress has been achieved in the construction of different variants of supersymmetric mechanics with $N=8$ supersymmetry. In [2] the structure of the sigma models with $N=4$ and $N=8$ supersymmetries has been analyzed. The first $N=8$ superconformally invariant action describing the low energy effective dynamics of a D0-brane has been constructed in [3] (see also [4, 5, 6]). Another version of $N=8$ superconformal mechanics with (**3,8,5**) and (**5,8,3**) $d=1$ supermultiplets was constructed in [7]. After a detailed investigation of the superfield structure of all possible $N=8, d=1$ supermultiplets in Ref. [8], new variants of $N=8$ supersymmetric mechanics with special Kähler geometry in the bosonic sector were developed in [9].

The variety of $N=8, d=1$ supermultiplets contains a very special one – i.e. the (**4,8,4**) supermultiplet. This supermultiplet may be constructed by joining two $N=4$ ones, namely (**4,4,0**) and (**0,4,4**) [7]. The supermultiplet (**4,4,0**) contains no auxiliary fields at all, while the (**0,4,4**) one does not include any physical bosons. The detailed discussion of the $N=4$ (**4,4,0**) supermultiplet has been performed in Ref. [10] within harmonic superspace. For $N=4$ supersymmetry this framework is not the most general one, because it puts additional constraint on the bosonic metric making the geometry strong HKT [2]. In the present paper we demonstrate that just this restriction is needed, in order to have $N=8$ supersymmetry. We construct the most general superfields action for the (**4,8,4**) supermultiplet. We denote this system as Supersymmetric Quaternionic Mechanics (SqM) because the physical bosons are represented just by quaternions q^{ia}. We show that it is possible to extend the action by a Fayet-Iliopoulos term, which results in the appearance of potential terms. We also provide a complete Hamiltonian description of the constructed system. Finally, we proposed a $N=9$ supermultiplet and prove that $N=9$ supersymmetry is too large to have an interesting action.

2 N=8 SqM: Lagrangian

In order to construct $N=8$ SqM with four bosonic and eight fermionic physical components we follow [8], introducing a real quartet of $N=8$ superfields $\mathcal{Q}^{ia}$ $\left((\mathcal{Q}^{ia})^\dagger=\mathcal{Q}_{ia}\right)$ depending on the coordinates of the $N=8$, $d=1$ superspace $\mathbb{R}^{(1|8)}$

$$\mathbb{R}^{(1|8)}=(t,\theta^{iA},\vartheta^{a\alpha}), \qquad \left(\theta^{iA}\right)^\dagger=\theta_{iA}, \qquad (\vartheta^{a\alpha})^\dagger=\vartheta_{a\alpha},$$

where i, a, A, $\alpha=1,2$ are doublet indices of four commuting $SU(2)$ subgroups of the automorphism group of $N=8$ Poincarè superalgebra[1], and obeying the constraints

$$D^{(i}_A\mathcal{Q}^{j)a}=0, \quad \nabla^{(a}_\alpha\mathcal{Q}^{b)i}=0\,. \tag{1}$$

The covariant spinor derivatives D^i_A, ∇^a_α are defined by

$$\begin{gathered}D^{iA}=\frac{\partial}{\partial\theta_{iA}}+i\theta^{iA}\partial_t\,, \quad \nabla^{a\alpha}=\frac{\partial}{\partial\vartheta_{a\alpha}}+i\theta^{a\alpha}\partial_t\,, \\ \left\{D^{iA},D^{jB}\right\}=2i\epsilon^{ij}\epsilon^{AB}\partial_t, \quad \left\{\nabla^{a\alpha},\nabla^{b\beta}\right\}=2i\epsilon^{ab}\epsilon^{\alpha\beta}\partial_t, \quad \left\{D^{iA},\nabla^{a\alpha}\right\}=0\,.\end{gathered} \tag{2}$$

Using (1), it is possible to show that the superfields $\mathcal{Q}^{ia}$ contain the following independent components:

$$q^{ia}=\mathcal{Q}^{ia}|, \quad \psi^{aA}=\frac{1}{2}D^A_i\mathcal{Q}^{ia}|, \quad \xi^{i\alpha}=\frac{1}{2}\nabla^\alpha_a\mathcal{Q}^{ia}|, \quad F^{\alpha A}=D^A_i\nabla^\alpha_a\mathcal{Q}^{ia}|\,, \tag{3}$$

where $|$ denotes the restriction to $\theta_{iA}=\vartheta_{a\alpha}=0$. Thus, we deal with the irreducible (**4,8,4**) supermultiplet.

[1]We use the following convention for the skew-symmetric tensor ϵ: $\epsilon_{ij}\epsilon^{jk}=\delta^k_i$, $\epsilon_{12}=\epsilon^{21}=1$.

In order to construct the corresponding action, we pass to $N=4$ superfields. As a first step, we single out a $N=4$ subspace in the $N=8$ superspace $\mathbb{R}^{(1|8)}$ as the set of coordinates

$$\mathbb{R}^{(1|4)} = (t, \theta_{iA}) \subset \mathbb{R}^{(1|8)} \tag{4}$$

and expand the $N=8$ superfields over the additional Grassmann coordinate $\vartheta_{a\alpha}$. Due to the following corollary of the constraints (1):

$$\nabla^{a\alpha}\nabla^{b\beta}\mathcal{Q}^{ic} = 2i\epsilon^{\alpha\beta}\epsilon^{cb}\dot{\mathcal{Q}}^{ia}, \tag{5}$$

one may easily see that in this expansion only the following four bosonic and four fermionic $N=4$ superfield projections:

$$q^{ia} = \mathcal{Q}^{ia}|_{\vartheta=0}, \quad \xi^{i\alpha} = \frac{1}{2}\nabla^{\alpha}_{a}\mathcal{Q}^{ia}|_{\vartheta=0} \tag{6}$$

are independent. These $N=4$ superfields are subjected, in virtue of eqs. (1), to the irreducibility constraints in $N=4$ superspace

$$D^{(i}_{A}q^{j)a} = 0, \quad D^{(i}_{A}\xi^{j)\alpha} = 0\,. \tag{7}$$

The implicit $N=4$ Poincaré supersymmetry transformations, completing the manifest one to the full $N=8$ supersymmetry, have the following form in terms of these $N=4$ superfields:

$$\delta q^{ia} = -\varepsilon^{a\alpha}\xi^{i}_{\alpha}, \quad \delta\xi^{i\alpha} = 2i\varepsilon^{a\alpha}\dot{q}^{i}_{a}. \tag{8}$$

The simplest way to deal with an action for the supermultiplet $\mathcal{Q}^{ia}$ is to use the harmonic superspace approach [11, 12, 10]. We use the definitions and conventions of Ref. [10]. The harmonic variables parameterizing the coset $SU(2)_R/U(1)_R$ are defined by the relations

$$u^{+i}u^{-}_{i} = 1 \quad \Leftrightarrow \quad u^{+}_{i}u^{-}_{j} - u^{+}_{j}u^{-}_{i} = \epsilon_{ij}\;, \;\; \overline{(u^{+i})} = u^{-}_{i}\;. \tag{9}$$

The harmonic projections of q^{ia} and $\xi^{i\alpha}$ are defined by

$$q^{+a} = q^{ia}u^{+}_{i}, \quad \xi^{+\alpha} = \xi^{i\alpha}u^{+}_{i}, \tag{10}$$

and the constraints (7) are rewritten as

$$D^{+A}q^{+a} = D^{+A}\xi^{+\alpha} = 0, \quad D^{++}q^{+a} = D^{++}\xi^{+\alpha} = 0\,. \tag{11}$$

Here $D^{\pm A} = D^{iA}u^{\pm}_{i}$ and $D^{\pm\pm} = u^{\pm i}\partial/\partial u^{\mp i}$ (in the central basis of the harmonic superspace), with D^{iA} given in (2).

The relations (11) imply that $q^{+a}, \xi^{+\alpha}$ are constrained analytic harmonic $N=4, d=1$ superfields living on the analytic subspace $(\zeta, u^{\pm}_{i}) \equiv (t_A, \theta^{+A}, u^{\pm}_{i})$ that is closed under $N=4$ supersymmetry. In this setting, the hidden $N=4$ supersymmetry transformations (8) are rewritten as

$$\delta q^{+a} = -\varepsilon^{a\alpha}\xi^{+}_{\alpha}, \quad \delta\xi^{+\alpha} = 2i\varepsilon^{a\alpha}\dot{q}^{+}_{a}. \tag{12}$$

The generic action has the form

$$S = \int du d\zeta^{--}\Big\{\mathcal{L}^{+a}(q^{+b},u)\,\partial_t\, q^{+}_{a} + \mathcal{L}(q^{+b},u)\,\xi^{+\alpha}\,\xi^{+}_{\alpha}\Big\} \tag{13}$$

where $\mathcal{L}^{+a}(q^{+b},u)$ and $\mathcal{L}(q^{+b},u)$ are, for the time being, arbitrary functions of q^{+a} and the harmonics u, and $dud\zeta^{--} = dudt_A d\theta^{+B}d\theta^{+}_{B}$ is the measure of integration over the analytic superspace. The action (13) is manifestly $N=4$ supersymmetric since it is written in terms of $N=4$ superfields. However, its invariance with respect to the hidden $N=4$ supersymmetry (12) must be explicitly checked. Varying the integrand in (13) with respect to (12) one may find that, in order to have hidden $N=4$ supersymmetry, one should impose the following condition:

$$\frac{\partial\mathcal{L}^{+a}}{\partial q^{+a}} = 4i\,\mathcal{L}\,. \tag{14}$$

The immediate corollary is that *any action* written in terms of the $N=4$ superfields q^{ia} in the *harmonic superspace* can be promoted to an invariant of $N=8$ supersymmetry by adding the interaction with

the superfields $\xi^{i\alpha}$. We would like to stress that the most general $N=4$ supersymmetric action for the superfields q^{ia} subjected to the constraints (7) should be written in the full $N=4$ superspace as

$$S=\int dtd^4\theta L(q)\ , \tag{15}$$

where $L(q)$ is an arbitrary scalar function. Requiring the $N=4$ supersymmetric action for q^{ia} to be representable in the analytic harmonic superspace imposes severe restrictions on the target-space geometry [10]. To clarify these restrictions and their consequences for the $N=8$ case let us pass to the components action. Integrating in (13) (with the constraints (14) imposed) over Grassmann variables and eliminating the auxiliary fields $F^{a\alpha}$ by their equations of motion, we end up with the following action:

$$\begin{aligned} S &= \int dt \left[G\left(\dot q^{ia}\dot q_{ia}+\frac{i}{2}\xi^{i\alpha}\dot\xi_{i\alpha}+\frac{i}{2}\psi^{aA}\dot\psi_{aA}\right)+\frac{i}{2}\frac{\partial G}{\partial q^{ia}}\left(\xi^{i\alpha}\xi^k_\alpha\,\dot q^a_k+\psi^{aA}\psi^b_A\,\dot q^i_b\right)+\right.\\ &\left.\frac{1}{8}\left(\frac{\partial^2 G}{\partial q^{ia}\partial q^{kb}}-2G^{-1}\frac{\partial G}{\partial q^{ia}}\frac{\partial G}{\partial q^{kb}}\right)\xi^{i\alpha}\xi^k_\alpha\,\psi^{aA}\psi^b_A\right]\ , \end{aligned} \tag{16}$$

where the metric is defined as

$$G=\int du\ \mathcal{L}(q^{+i},u)\big|_{\theta=0}\ . \tag{17}$$

It follows immediately from (17) that such a metric G satisfies the condition [10]

$$\triangle G\equiv\frac{\partial^2}{\partial q^{ia}\partial q_{ia}}\ G=0\ . \tag{18}$$

This type of metrics defines a strong HKT geometry [2].

It is worth pointing out that the restriction (17) in the $N=4$ case makes the theory almost trivial. Indeed, passing to the components in the most general action (15) we get

$$S_{N=4}=\int dt\left[G\left(\dot q^{ia}\dot q_{ia}+\frac{i}{2}\psi^{aA}\dot\psi_{aA}\right)+\frac{i}{2}\frac{\partial G}{\partial q^{ia}}\psi^{aA}\psi^b_A\ \dot q^i_b-\frac{1}{3}\triangle G\,\psi^{aA}\psi^b_A\psi^B_a\psi_{Bb}\right], \tag{19}$$

where

$$G=\frac{\partial^2}{\partial q^{ia}\partial q_{ia}}L(q)\ . \tag{20}$$

Now, it is evident that the constraint (18), being imposed, kills the four-fermion interaction term in the action. Correspondingly, the Hamiltonian for such a theory does not contain fermions at all and describes the pure bosonic sigma-model. In the $N=8$ SqM case the action (16) still contains a four-fermion term which combines the fermions coming from two $N=4$ supermultiplets q^{ia} and $\xi^{i\alpha}$.

Thus, we conclude that the action (16) is invariant with respect to $N=8$ supersymmetry, which is realized on the physical component fields as follows:

$$\delta q^{ia}=-\epsilon^{iA}\psi^a_A-\varepsilon^{a\alpha}\xi^i_\alpha\ ,\quad \delta\psi^{aA}=2i\epsilon^{iA}\dot q^a_i-\varepsilon^{a\alpha}F^A_\alpha\ ,\quad \delta\xi^{i\alpha}=\epsilon^{iA}F^\alpha_A+2i\varepsilon^{a\alpha}\dot q^i_a\ , \tag{21}$$

with ϵ^{iA}, $\varepsilon^{a\alpha}$ being the parameters of two $N=4$ supersymmetries acting on θ^{iA} and $\vartheta^{a\alpha}$, respectively, and with the auxiliary fields $F_{\alpha A}$ defined as

$$F_{\alpha A}=-G^{-1}\frac{\partial G}{\partial q^{ia}}\xi^i_\alpha\psi^a_A\ . \tag{22}$$

Using the Noether theorem one can find classical expressions for the conserved supercharges corresponding to the supersymmetry transformations (21)

$$Q^{iA}=G\,\dot q^i_a\,\psi^{aA}+\frac{i}{3}\frac{\partial G}{\partial q^b_i}\,\psi^{Ac}\,\psi_{cD}\,\psi^{bD}\ ,\quad S^{a\alpha}=G\,\dot q^a_i\,\xi^{i\alpha}+\frac{i}{3}\frac{\partial G}{\partial q^k_a}\,\xi^\alpha_j\,\xi^{j\beta}\,\xi^k_\beta\ . \tag{23}$$

We conclude this section with a few comments.

Firstly, we succeeded in the construction of a $N=8$ supersymmetric action for the (**4,8,4**) supermultiplet (16) with the bosonic metric G subjected to the constraint (18). This restriction appears naturally

in such consideration because the model formulates in the harmonic superspace. In the next Section we will demonstrate that this constraint is not an *artifact* of this approach, but it is an unavoidable feature of $N=8$ SqM.

Secondly, the action (13) may be extended by the Fayet-Iliopoulos term

$$\widetilde{S} = S + \int dud\zeta^{--}\, \lambda_{A\alpha}\, \theta^{+A}\, \xi^{+\alpha}\,, \tag{24}$$

with constant real matrix parameters $\lambda^{A\alpha}$ $\left(\left(\lambda^{A\alpha}\right)^{\dagger} = \lambda_{A\alpha}\right)$. As a result, the component action gets the potential terms

$$S_p = \frac{1}{16}\int dt \left[\frac{\lambda^2}{G} + 4\, G^{-1}\, \frac{\partial G}{\partial q^{ia}}\, \lambda_{A\alpha}\, \xi^{i\alpha}\, \psi^{aA}\right]. \tag{25}$$

Such a possibility to get potential terms in the joint action for (**4,4,0**) and (**0,4,4**) supermultiplets has been noted in Ref. [10].

A final comment concerns the $N=8$ superconformal invariant action. From the considerations in Refs. [7, 8] it follows that the corresponding $N=8$ superconformal group should be the $OSp(4^{\star}|4)$ one. As an additional requirement, the bosonic metric G should be invariant with respect to all automorphism transformations, which means $G = G(q^2)$. Thus, as a solution of the constraint (18), we have

$$G_{conf} = \frac{a}{q^2} + b, \quad a, b = \text{const}. \tag{26}$$

The solution with $a \neq 0$ explicitly breaks even dilatation invariance. So, we conclude that the unique candidate to be a $N=8$ superconformal action is a free action with constant metric G.

3 N=8 SqM: Hamiltonian

In order to find the classical Hamiltonian, we define the momenta p_{ia}, $\pi^{(\psi)}_{aA}$, $\pi^{(\xi)}_{a\alpha}$ conjugated to q^{ia}, ψ^{aA} and $\xi^{i\alpha}$, respectively, as

$$p_{ia} = 2\, G\, \dot q_{ia} - \frac{i}{2}\frac{\partial G}{\partial q^{ib}}\psi^{aA}\psi^{b}_{A} - \frac{i}{2}\frac{\partial G}{\partial q^{ja}}\xi^{i\alpha}\xi^{j}_{\alpha}\,, \quad \pi^{(\psi)}_{aA} = -\frac{i}{2}\, G\, \psi_{aA},\; \pi^{(\xi)}_{i\alpha} = -\frac{i}{2}\, G\, \xi_{i\alpha} \tag{27}$$

and introduce the canonical Poisson brackets

$$\left\{q^{ia}, p_{jb}\right\} = \delta^{i}_{j}\delta^{a}_{b}, \quad \left\{\psi^{aA}, \pi^{(\psi)}_{bB}\right\} = -\delta^{a}_{b}\delta^{A}_{B}, \quad \left\{\xi^{i\alpha}, \pi^{(\psi)}_{j\beta}\right\} = -\delta^{\alpha}_{\beta}\delta^{i}_{j}. \tag{28}$$

From the explicit form of the fermionic momenta (27) it follows that the system possesses second–class constraints. Using the standard procedure, we get the following Dirac brackets for the canonical variables[2]
:

$$\begin{aligned}
&\left\{q^{ia}, p_{jb}\right\} = \delta^{i}_{j}\delta^{a}_{b}, \quad \left\{\psi^{aA}, \pi^{(\psi)}_{bB}\right\} = -\delta^{a}_{b}\delta^{A}_{B}, \quad \left\{\xi^{i\alpha}, \pi^{(\psi)}_{j\beta}\right\} = -\delta^{\alpha}_{\beta}\delta^{i}_{j}\,,\\
&\{\tilde p^{ia}, \tilde p^{jb}\} = -\frac{i}{2}\,\epsilon^{ab} R^{i}{}_{c}{}^{j}{}_{d}\, \psi^{cA}\psi^{d}_{A} - \frac{i}{2}\,\epsilon^{ij}\, R_{m}{}^{a}{}_{n}{}^{b}\, \xi^{m\alpha}\xi^{n}_{\alpha}\\
&\{\tilde p^{ia}, \psi_{bA}\} = \delta^{a}_{b}\, \Gamma^{id}\, \psi_{dA}\,, \quad \{\tilde p^{ia}, \xi_{k\beta}\} = \delta^{i}_{k}\, \Gamma^{la}\, \xi_{l\beta}\,,\\
&\{\psi^{aA}, \psi^{bB}\} = -\frac{i}{G}\,\epsilon^{ab}\,\epsilon^{AB}\,, \quad \{\xi^{i\alpha}, \xi^{k\beta}\} = -\frac{i}{G}\,\epsilon^{ik}\,\epsilon^{\alpha\beta}\,,
\end{aligned} \tag{29}$$

where

$$R_{iajb} = \frac{\partial^2 G}{\partial q^{ia}\partial q^{jb}} - \frac{2}{G}\frac{\partial G}{\partial q^{ia}}\frac{\partial G}{\partial q^{jb}}, \quad \Gamma_{i\,a} = \frac{1}{2}\,\frac{\partial \ln G}{\partial q^{i\,a}} \tag{30}$$

and the bosonic momenta $\tilde p^{ia}$ have been defined as

$$\tilde p^{i\,a} = p^{i\,a} + i\, G\, \left(\Gamma^{a}_{k}\, \xi^{i\alpha}\xi^{k}_{\alpha} + \Gamma^{i}_{b}\, \psi^{aA}\psi^{b}_{A}\right). \tag{31}$$

[2]From now on, the symbol $\{\,,\}$ stands for the Dirac bracket.

Now one can check that the supercharges Q_{iA}, $S_{a\alpha}$ (23), being rewritten through the momenta $\tilde{p}^{ia}$ as

$$Q^{iA} = \tilde{p}^i_a\, \psi^{aA} + \frac{i}{3}\frac{\partial G}{\partial q^b_i}\, \psi^{Ac}\, \psi_{cD}\, \psi^{bD} \;, \qquad S^{a\alpha} = \tilde{p}^a_i\, \xi^{i\alpha} + \frac{i}{3}\frac{\partial G}{\partial q^k_a}\, \xi^\alpha_j\, \xi^{j\beta}\, \xi^k_\beta \;, \tag{32}$$

and the Hamiltonian

$$H = \frac{\tilde{p}^2}{2G} - \frac{1}{4} R_{iajb}\, \psi^{aA}\psi^b_A\, \xi^{i\alpha}\xi^j_\alpha \tag{33}$$

form the standard $N=8$ superalgebra:

$$\{Q^{iA}, Q^{kB}\} = -i\, \epsilon^{ik}\, \epsilon^{AB}\, H \;, \quad \{S^{a\alpha}, S^{b\beta}\} = -i\, \epsilon^{ab}\, \epsilon^{\alpha\beta}\, H \;, \quad \{Q^{iA}, S^{a\alpha}\} = 0 \,. \tag{34}$$

With these, we completed the classical description of $N=8$ SqM. Before closing this Section, let us go back and clarify the necessities of the constraint (18). The idea is rather simple. The Dirac brackets (29) provide a quite general basis to construct the system without any constraints on the metric. The most general Ansatz for the supercharges Q^{iA} and $S^{a\alpha}$ reads

$$Q^{iA} = \tilde{p}^i_a\, \psi^{aA} + f^i_a\, \psi^{aB}\psi_{dB}\psi^{dA} + g_{ka}\, \psi^{aA}\, \xi^{k\alpha}\xi^i_\alpha \,, \; S^{a\alpha} = \tilde{p}^a_i\, \xi^{i\alpha} + t^a_i\, \xi^i_\beta\xi^{k\alpha}\xi^\beta_k + h_{ib}\, \xi^{i\alpha}\psi^{aA}\psi^b_A \;, \tag{35}$$

where f^i_a, g_{ka}, t^a_i and h_{ib} are arbitrary functions of q^{ia}. Straightforward but rather lengthly calculations show that the supercharges (35) obey the superalgebra (34) only if they have form as in (32) and if the metric G obeys (18). So, the main restriction on the metric (18) is an unavoidable feature of $N=8$ SqM.

Finally, let us note that the generalization of $N=8$ SqM to the n-dimensional case is straightforward and will be presented elsewhere.

4 Towards N=9 supersymmetry

One of the interesting features of the $N=8$ supermultiplets with (**4,8,4**) content is the possibility to realize one additional supersymmetry on a pair of such multiplets. This can be done as follows. Let us start with a pair of $N=8$ supermultiplets $\mathcal{Q}^{ia}$ and $\widehat{\mathcal{Q}}^{i\alpha}$ subject to the constraints

$$D^{(i}_A \mathcal{Q}^{j)a} = 0, \quad \nabla^{(a}_\alpha \mathcal{Q}^{b)i} = 0 \;, \qquad D^{(i}_A \widehat{\mathcal{Q}}^{j)\alpha} = 0, \quad \nabla^{(\alpha}_a \widehat{\mathcal{Q}}^{\beta)i} = 0 \;. \tag{36}$$

In the $N=4$ subspace of the $N=8$ superspace $\mathbb{R}^{(1|4)}$ (4) only the following eight bosonic and eight fermionic $N=4$ superfields projections:

$$q^{ia} = \mathcal{Q}^{ia}|_{\vartheta=0}, \quad \xi^{i\alpha} = \frac{1}{2}\nabla^\alpha_a \mathcal{Q}^{ia}|_{\vartheta=0}, \qquad \hat{q}^{i\alpha} = \widehat{\mathcal{Q}}^{i\alpha}|_{\vartheta=0}, \quad \hat{\xi}^{ia} = \frac{1}{2}\nabla^a_\alpha \widehat{\mathcal{Q}}^{i\alpha}|_{\vartheta=0}, \tag{37}$$

are independent. These $N=4$ superfields are subject, in virtue of eqs. (36), to the irreducibility constraints in $N=4$ superspace

$$D^{(i}_A q^{j)a} = 0, \; D^{(i}_A \xi^{j)\alpha} = 0 \,, \qquad D^{(i}_A \hat{q}^{j)\alpha} = 0, \; D^{(i}_A \hat{\xi}^{j)a} = 0 \,. \tag{38}$$

The transformations under implicit $N=4$ supersymmetry have the form

$$\delta q^{ia} = -\eta^{a\alpha}\xi^i_\alpha, \; \delta\xi^{i\alpha} = 2i\eta^{a\alpha}\dot{q}^i_a, \quad \delta\hat{q}^{i\alpha} = -\eta^{a\alpha}\hat{\xi}^i_a, \; \delta\hat{\xi}^{ia} = 2i\eta^{a\alpha}\dot{\hat{q}}^i_\alpha, \tag{39}$$

while the additional, ninth supersymmetry may be realized on these $N=4$ superfields as

$$\delta q^{ia} = -\epsilon\, \hat{\xi}^{ia} \,, \quad \delta\xi^{a\alpha} = -2i\,\epsilon\, \partial_t \hat{q}^{a\alpha} \,, \qquad \delta\hat{q}^{a\alpha} = -\epsilon\, \xi^{a\alpha} \,, \quad \delta\hat{\xi}^{ia} = -2i\,\epsilon\, \partial_t\, q^{ia} \,. \tag{40}$$

Thus, we have a $N=9$ supermultiplet with (**8,16,8**) components in full agreement with the considerations in Refs. [13], where it has been proven that the minimal $N=9$ supersymmetric multiplet must have 16 bosonic and 16 fermionic fields.

As in the case of one $N=8$ supermultiplet $\mathcal{Q}^{ia}$ we considered in the previous Sections, the harmonic superspace provides the most adequate setup for constructing the action. After introducing harmonic projections of $q^{ia}, \xi^{i\alpha}, \hat{q}^{i\alpha}$ and $\hat{\xi}^{ia}$ by

$$q^{+a} = q^{ia}u^+_i, \; \xi^{+\alpha} = \xi^{i\alpha}u^+_i, \qquad \hat{q}^{+\alpha} = \hat{q}^{i\alpha}u^+_i, \; \hat{\xi}^{+a} = \hat{\xi}^{ia}u^+_i, \tag{41}$$

one may write the most general Ansatz for the hypothetical $N = 9$ supersymmetric action

$$
\begin{aligned}
S_{N=9} &= \int du d\zeta^{--} \Big\{ \mathcal{L}^{+a}(q^{+b}, \hat{q}^{+\beta}, u)\, \partial_t\, q^+_a + \mathcal{L}^{+\alpha}(q^{+b}, \hat{q}^{+\beta}, u)\, \partial_t\, \hat{q}^+_\alpha \\
&+ \mathcal{L}(q^{+b}, \hat{q}^{+\beta}, u)\, \xi^{+\alpha}\, \xi^+_\alpha + \tilde{\mathcal{L}}(q^{+b}, \hat{q}^{+\beta}, u)\, \hat{\xi}^{+a}\, \hat{\xi}^+_a + \mathcal{L}_{\alpha a}(q^{+b}, \hat{q}^{+\beta}, u)\, \xi^{+\alpha}\, \hat{\xi}^{+a} \Big\}, \qquad (42)
\end{aligned}
$$

where $\mathcal{L}^{+a}, \mathcal{L}^{+\alpha}, \mathcal{L}, \tilde{\mathcal{L}}, \mathcal{L}_{\alpha a}$ are arbitrary functions of $N = 4$ superfields $q^{+a}, \hat{q}^{+\alpha}$ and harmonics $u^\pm_i$.

Being written in $N = 4$ superspace, the action (42) enjoys manifest $N = 4$ supersymmetry while the invariance with respect to implicit $N = 4$ (39) and ninth supersymmetry (40) should be checked. The results of rather length calculations are

- **N=8**: the invariance with respect to implicit $N = 4$ supersymmetry (39) and, therefore the $N = 8$ invariance of the action put the following restrictions:

$$
\frac{\partial \mathcal{L}^{+a}}{\partial \hat{q}^{+\alpha}} = \frac{\partial \mathcal{L}}{\partial \hat{q}^{+\alpha}} = 0, \quad \frac{\partial \mathcal{L}^{+\alpha}}{\partial q^{+a}} = \frac{\partial \tilde{\mathcal{L}}}{\partial q^{+a}} = 0, \quad \mathcal{L}_{\alpha a} = 0, \qquad (43)
$$

$$
\frac{\partial \mathcal{L}^{+a}}{\partial q^{+a}} = 4i\, \mathcal{L}\,, \quad \frac{\partial \mathcal{L}^{+\alpha}}{\partial \hat{q}^{+\alpha}} = 4i\, \tilde{\mathcal{L}}\,. \qquad (44)
$$

  Thus, $N = 8$ supersymmetry forbids the interaction between two supermultiplets $q^{ia}, \xi^{i\alpha}$ and $\hat{q}^{i\alpha}, \hat{\xi}^{ia}$. This result is not so strange and may be treated as a dimensionally reduced version of the results obtained in Ref. [14], where possible interactions of hypermultiplets in $d = 2$ have been analyzed.

- **N=9**: the ninths supersymmetry makes the action completely free

$$
\mathcal{L}^{+a} = q^{+a}, \quad \mathcal{L}^{+\alpha} = \hat{q}^{+\alpha}. \qquad (45)
$$

Therefore, we conclude that $N = 9$ supersymmetry is so restrictive that only free actions can enjoy it. These results may be considered as an indirect proof (not complete, of course) of the statement that the theories with eight supercharges are the highest N theories with extended supersymmetries which have a rich geometric structure of the target space (see e.g. [1]).

5 Conclusions

In this paper we constructed a new version of $N = 8$ supersymmetric mechanics based on the off-shell multiplet (**4,8,4**). We showed that the most general action for this supermultiplet can be constructed within harmonic superspace. We extended the action by a Fayet-Iliopoulos term, which gives rise to potential terms. We also provide a detailed Hamiltonian description of the constructed system.

The supermultiplet (**4,8,4**) is rather unique, because it may be decomposed into the two very special $N = 4$ supermultiplets (**4,4,0**) and (**0,4,4**), which cannot exist in higher dimensions. As an interesting result, we get the restriction on the bosonic metric, which should obey the Laplace equation. This restriction, which is automatically satisfied in the harmonic superspace approach for the (**4,4,0**) supermultiplet [10] and appears to be too strong in $N = 4$ supersymmetric mechanics, is unavoidable in the case of $N = 8$ supersymmetry. Combining two (**4,8,4**) supermultiplets we constructed a $N = 9$ supermultiplet. Unfortunately, $N = 9$ supersymmetry is too large and restricts the most general action to be a free one.

Apart form the considered decomposition of the $N = 8$ supermultiplet (**4,8,4**) into (**4,4,0**) and (**0,4,4**), there is another one (**4,8,4**)=(**2,4,2**)+(**2,4,2**) [8]. It seems interesting to find appropriate Lagrangian and Hamiltonian descriptions of $N = 8$ SqM for such a splitting, especially due to the existence of a complete Hamiltonian analysis of the $N = 4$ supersymmetric mechanics with 4n+4n phase space [15].

An obvious project for future study is to formulate $N = 8$ SqM in the $N = 8$ harmonic superspace with double sets of harmonics, similarly to Ref. [16]. Another possible extension of the proposed system is related with the relaxing of the constraints for the (**0,4,4**) supermultiplet by admitting a constant part for the auxiliary fields. As a result, in such a version we expect the appearance of central charges in the $N = 8$ super Poincaré algebra.

Acknowledgements

We wish to thank Armen Nersessian for many enlightening comments and discussions.

This research was partially supported by the European Community's Marie Curie Research Training Network under contract MRTN-CT-2004-005104 Forces Universe, INTAS-00-00254 grant, RFBR-DFG grant No 02-02-04002, grant DFG No 436 RUS 113/669, and RFBR grant No 03-02-17440. S.K. thanks INFN — Laboratori Nazionali di Frascati for the warm hospitality extended to him during the course of this work.

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